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Optimal distribution of material properties for an elastic continuum with structure-dependent body force

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Abstract

The simultaneous optimization of material properties and structural layout for an elastic continuum is formulated and analyzed. The objective is to obtain the maximum structural stiffness for prescribed surface loads and displacements, taking into account a body force that depends on the structural layout. Optimization with an account of self-weight or centrifugal forces are examples of these type of problems. Arbitrary elasticity tensor fields are considered as the problem's variable and necessary conditions satisfied by the solution are established. The use of a spectral decomposition of the elasticity tensor is emphasized since it provides a simple geometrical interpretation. Typical examples which illustrate the effects of the structure-dependent body force are analyzed. It is found that a commonly used isoperimetric restriction (known as the resource constraint) is not necessarily active and that the optimal structure is locally stiffer in areas where the body force and the displacement field have opposite directions and locally weaker otherwise. This, interestingly, leads to a non-symmetrical distribution of material properties even for prismatic bodies under anti-symmetric surface loads. © 1999 Elsevier Science Ltd. All rights reserved.

1. Introduction

The problem of characterizing the maximum structural stiffness of a linearly elastic continuum structure has been extensively analyzed using different techniques. It has been recognized that it is possible to find simultaneously the optimal material properties and structural layout. One way to achieve this is simply by enlarging the space of variables in order to include all possible structures, i.e., by considering completely general elasticity tensor fields. This approach is known as the free-material optimization method and it relies on the use of elasticity tensor fields which, a priori, correspond to anisotropic and inhomogeneous materials (Bendsøe et al., 1994). Other methods consider a special class of anisotropic materials, namely composites obtained by combining two distinct homogeneous

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materials, although the problem requires relaxation in order to include optimal composites. In that case, usually assuming the existence of some type of periodic microgeometry, homogenization theory is used to compute the effective properties at a macroscopic level. Parameters related to the microgeometry are used as variables (Bendsøe, 1995 for a comprehensive review of both methods). The advantage of the free-material optimization method is that issues related to homogenization do not need to be taken into account. Once the optimal material properties have been identified (at a macroscopic level) one can solve, if desired, an inverse problem in order to obtain at each point a microgeometry that provides the best match to the optimal properties as shown by Sigmund (1994) (see also Milton and Cherkaev, 1995). The purpose of the present analysis is to include, within the free-material formulation, a body force that depends on the optimization variable (i.e., on the elasticity tensor). The motivation for this extension is to analyze cases where a non-negligible inertial force acts on the structure. Typical examples are structural elements rotating at a relatively large angular speed or large structures whose own weight becomes a relevant factor in the analysis. Early work on problems with a structure-dependent body force include applications where the objective is to distribute a given amount of material in an elastic structure in order to match a desired natural frequency (Prager, 1974). A more recent review on the subject was given by Olhoff (1987). Here, however, the objective is to maximize the structural stiffness taking into account a variable body force but no restrictions are placed upon natural frequencies.

The use of a formulation based on a spectral decomposition of the elasticity tensor is emphasized since it provides a simple geometrical interpretation of the results in a fourth-order tensor space. It will be shown that the optimized material has two main components: a term which provides the optimal stiffness and a term which provides the required stability against possible perturbations of the prescribed loads. In this analysis, it is assumed that the magnitude of the elastic moduli and the material's mass density are correlated. Even though in general there is no a priori relation between these quantities, some models used for cellular solids exhibit a functional relation between the elastic moduli (Gibson and Ashby, 1997). From a different point of view, a relation between mass density and elastic properties can be assumed if one has in mind using, a posteriori, a method like the one proposed by Sigmund (1994). In that case, a microgeometry composed of weak (essentially void) and strong materials is used to match given effective properties. Bearing in mind that the strong material is fixed (and so is its mass density), the resulting volume fraction of the strong material is directly related to the mass density of the composite material. Thus, in this restricted sense, one can establish a relation between elastic moduli and mass density and it is assumed that this relation is strictly monotonic.

The outline of the paper is as follows: in Section 2, the optimization problem is formulated and under some specific assumptions a simplified version of it is derived by analyzing the restrictions imposed at a local level (for a continuum) by the global structural requirements. Optimality conditions for the problem are developed in Section 3 and specific examples with an inertial body force are treated in Section 4. A discussion and concluding remarks follow in the last section.

2. Formulation of the problem

2.1. Notation

As a general scheme of notation, scalar quantities are represented by italicized normal-face letters, vectors and points in a three-dimensional Euclidean space by bold-face lower case letters (except for the stress tensor $\boldsymbol{\sigma}$ and strain tensor $\boldsymbol{\varepsilon}$), second order tensors by bold-face upper case letters and fourth order tensors by bold-face upper case italicized letters. Throughout this analysis, coordinate-free notation is employed. To ease the transition to indicial notation, the following definitions are referred to a three-dimensional Cartesian basis (indices range in $\{1, 2, 3\}$, δ_{ij} is Kronecker's delta and the summation convention is used):

$$\mathbf{a}\cdot\mathbf{b} = a_i b_i, \quad \mathbf{A}\cdot\mathbf{B} = A_{ij}B_{ij}, \quad \boldsymbol{C}\cdot\boldsymbol{D} = C_{ijkl}D_{ijkl}, \|\mathbf{A}\| = \sqrt{\mathbf{A}\cdot\mathbf{A}}, \quad \|\boldsymbol{C}\| = \sqrt{\boldsymbol{C}\cdot\boldsymbol{C}},$$

$$(\mathbf{Ab})_i = A_{ij}b_j, \quad (\mathbf{AB})_{ij} = A_{ik}B_{kj}, \quad (\boldsymbol{C}\mathbf{A})_{ij} = C_{ijkl}A_{kl}, \quad (\mathbf{a}\otimes\mathbf{b})_{ij} = a_i b_j, \quad (\mathbf{A}\otimes\mathbf{B})_{ijkl} = A_{ij}B_{kl},$$

$$(\mathbf{I})_{ij} = \delta_{ij}, \quad (\boldsymbol{I})_{ijkl} = \tfrac{1}{2}(\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk}).$$

In general, when the meaning is clear by the context, a scalar, vector or tensor field defined in a domain Ω and its value at a point $\mathbf{x}$ will be denoted by the same letter, e.g., $\mathbf{A} = \mathbf{A}(\mathbf{x})$. Similarly, a scalar, vector or tensor function of a scalar, vector of tensor field will be denoted by the same symbols, e.g., $\varphi = \varphi(\mathbf{A}) = \varphi(\mathbf{A}(\mathbf{x}))$, when it is clear by the context that φ is not, for example, a functional.

2.2. Preliminaries

Consider a linearly elastic body that occupies a given domain Ω. Suppose that the prescribed tractions $\mathbf{t}$ and displacements $\mathbf{u}$ are

$$\left.\begin{aligned} \mathbf{t}(\mathbf{x}) &= \hat{\mathbf{t}}(\mathbf{x}) \quad \mathbf{x}\in\partial\Omega_t \\ \mathbf{u}(\mathbf{x}) &= 0 \quad\quad \mathbf{x}\in\partial\Omega_u \end{aligned}\right\} \tag{1}$$

where $\partial\Omega_t \bigcup \Omega_u = \partial\Omega$ is the boundary of Ω. Only mixed boundary conditions of the form of eqn (1) are considered here. Let $\boldsymbol{\varepsilon}$ be the strain tensor field which, viewed as a function of $\mathbf{u}(\mathbf{x})$, is given by

$$\boldsymbol{\varepsilon} = \varepsilon(\mathbf{u}) = \boldsymbol{\varepsilon}(\mathbf{u}(\mathbf{x})) = \tfrac{1}{2}(\nabla\mathbf{u}(\mathbf{x}) + \nabla\mathbf{u}(\mathbf{x})^T), \quad \forall\mathbf{x}\in\Omega.$$

Let $\boldsymbol{C}$ be a (fourth-order) elasticity tensor field. The stress tensor field $\boldsymbol{\sigma}$ is related to the strain tensor via the constitutive law

$$\boldsymbol{\sigma}(\mathbf{x}) = \boldsymbol{C}(\mathbf{x})\boldsymbol{\varepsilon}(\mathbf{x}).$$

As a matter of terminology, the word 'structure' should be understood in the present analysis to refer to a distribution of material properties over the given domain Ω, i.e., to the field $\boldsymbol{C}(\mathbf{x})$, $\mathbf{x}\in\Omega$. By 'optimal distribution' it is meant that a specific structure (i.e., distribution of $\boldsymbol{C}(\mathbf{x})$) is sought based on maximization of the overall stiffness of the structure. A formal statement of this problem is given in Section 2.4.

To obtain the optimal distribution of material properties, the basic idea is to let the elasticity tensor field be variable. It is worth noting that one can accomplish two seemingly distinct objectives with this approach. Firstly, the minimizing field $\boldsymbol{C}_0$ provides information on the mechanical properties of a material at each point (i.e., the material optimization). Secondly, since points where the material properties are 'small' are interpreted as holes (in a sense to be defined later), then $\boldsymbol{C}_0$ also provides the shape of the structure within the domain Ω by specifying the location of holes.

Hence, the shape optimization (structural layout) is obtained as a by-product of the material optimization. This ability to perform simultaneously optimization of material properties and structural layout is one of the main benefits of the free-material optimization method and its simplicity (compared to homogenization techniques) provides an additional advantage.

2.3. Body force and space of admissible variables

The space of admissible optimization variables is, in this case, a subspace of fourth-order tensor fields defined in Ω. For practical reasons, both physical and mathematical, some additional restrictions on the space of optimization variables are required. To introduce these restrictions it is convenient to use a spectral decomposition of the elasticity tensor. Since the admissible elasticity tensors are symmetric (major symmetry), they can be represented at each point $\mathbf{x}$ in terms of their real eigenvalues α_μ ($\mu = 1, \ldots, 6$) and unit eigentensors $\mathbf{A}_\mu$ (2-tensors), i.e., the spectral decomposition is given by

$$\boldsymbol{C} = \sum_{\mu=1}^{6} \alpha_\mu \mathbf{A}_\mu \otimes \mathbf{A}_\mu, \tag{2}$$

where $\alpha > 0$ and $\{\mathbf{A}_\mu\}_{\mu=1}^{6}$ forms an orthonormal basis for the space of symmetric 2-tensors. Recall that for a general (anisotropic) material, the eigentensors contain information about the material behavior (via 'angles' in a tensor space) but the "true" stiffness information is provided by the six principal values of the elastic moduli (i.e., eigenvalues α_μ). For example, for an isotropic material there are two distinct eigenvalues: 3κ (multiplicity one) and 2μ (multiplicity five), where κ is the bulk modulus and μ is the shear modulus (Knowles, 1995). It is also important to note that the result of this decomposition is that the design variables are now explicitly the eigenvalues and eigentensors.

The fact that the eigenvalues are strictly positive means that $\boldsymbol{C}$ is positive definite. To satisfy this requirement, it is assumed that all eigenvalues are bounded below by the same constant α_m. Additionally, an upper bound α_M is imposed on each α_μ (same constant α_M for all α_μ). Since $\{\alpha_\mu\}_{\mu=1}^{6}$ are the principal stiffnesses, the upper bound is seen as a limitation of real materials: the elastic moduli cannot exceed the values of the stiffest material known in nature (or, in fact, of a predetermined material that would be used to reinforce the structure). Finally, a global upper bound in the eigenvalues (known as resource constraint) is also imposed. To this end, consider a positive-valued scalar function $\bar{\varphi}$ of $\boldsymbol{C}$ which is referred to as the resource constraint density. In the work of Bendsøe et al. (1994), a resource constraint density equal to the trace of $\boldsymbol{C}$ was used, i.e.,

$$\bar{\varphi}(\boldsymbol{C}) = \sum_{\mu=1}^{6} \alpha_\mu.$$

Following a similar approach, the resource constraint density considered here is assumed to be independent of the orientation of the principal directions of the elasticity tensor and dependent on the eigenvalues of $\boldsymbol{C}$ via their maximum only, i.e.,

$$\bar{\varphi}(\boldsymbol{C}) = \alpha, \tag{3}$$

where α is the maximum principal stiffness, i.e.

$$\alpha = \alpha(\mathbf{x}) = \max_{1 \leqslant \mu \leqslant 6} \{\alpha_\mu(\mathbf{x})\}. \tag{4}$$

Observe that eqn (3) is not the only choice since $\bar{\varphi}$ could depend on $\mathbf{A}_\mu$ (the orientation of material properties) which could be interpreted, e.g., as a constraint imposed by manufacturing requirements. In terms of α, the resource constraint is expressed by the following isoperimetric inequality:

$$\int_\Omega \alpha(\mathbf{x}) \mathrm{d}v \leqslant R,$$

where the left hand side is the total resource and the given positive constant R represents an upper bound on the resource. The main role of the resource constraint is to rule out some trivial solutions. A typical trivial case occurs when the body force $\mathbf{b}$ is zero and the isoperimetric inequality is not enforced: the stiffest structure is obtained when the optimization variable coincides with the local upper bound, i.e., it coincides with the stiffest material properties everywhere.

For future use, all the requirements on the space of admissible optimization variables can be collected via the following sets:

$$\mathscr{S}_{\alpha_\mu} = \left\{ \{\alpha\}_{\mu=1}^{6} \,|\, \alpha_m \leqslant \alpha_\mu(\mathbf{x}) \leqslant \alpha_M, \int_\Omega \alpha \, \mathrm{d}v \leqslant R, \forall \mathbf{x} \in \Omega \right\},$$

and

$$\mathscr{S}_{\mathbf{A}_\mu} = \left\{ \{\mathbf{A}_\mu\}_{\mu=1}^{6} \,|\, \mathbf{A}_\mu \cdot \mathbf{A}_\nu = \delta_{\mu\nu}, \mathbf{A}_\mu = \mathbf{A}_\mu^T, \sum_{\mu=1}^{6} \mathbf{A}_\mu \otimes \mathbf{A}_\mu = \boldsymbol{I} \right\},$$

where α is defined by eqn (4) and $\delta_{\mu\nu}$ is Kronecker's delta. The symmetry of each $\mathbf{A}_\mu$ reflects the minor symmetries of $\boldsymbol{C}$. Now, it is assumed that there exist functional relations between the principal elastic moduli (eigenvalues α_μ) and the mass density ρ, i.e., $\alpha_\mu = \bar{\alpha}_\mu(\rho)$. To motivate this assumption from a physical point of view, one could refer, for example, to models used for cellular solids (Gibson and Ashby, 1997). In that context it has been found experimentally that, in the linearly elastic range, the material properties scale with some power of the mass density (e.g., the shear modulus for elastomeric foams is $\mu \sim \rho^2$, etc.). Similar relations are sometimes used in some simple models for composite materials where the material properties depend on the volume fraction of each component (hence they depend on the relative mass densities). Furthermore, assuming that these relations can be inverted, then the mass density could be expressed as a function of one of the eigenvalues of $\boldsymbol{C}$; in particular, $\rho = \bar{\rho}(\alpha)$, where α is the maximum principal stiffness. Thus, assuming the existence of such a model, an (inertial) body force can be viewed as a function of α, i.e.,

$$\mathbf{b} = \hat{\mathbf{b}}(\bar{\rho}(\alpha)) = \bar{\mathbf{b}}(\alpha). \tag{5}$$

With this interpretation, the resource constraint can also be seen as a restriction on the total mass of the structure. However, it is important to mention that the present analysis is limited to cases where a relation such as (5) can be identified.

Let W be the work done by the external forces, i.e.,

$$W[\mathbf{u},\bar{\mathbf{b}}(\alpha)] = \int_{\Omega} \bar{\mathbf{b}}(\alpha(\mathbf{x}))\cdot\mathbf{u}(\mathbf{x})\,\mathrm{d}v + \int_{\partial\Omega_t} \hat{\mathbf{t}}(\mathbf{x})\cdot\mathbf{u}(\mathbf{x})\,\mathrm{d}a. \tag{6}$$

Let $\mathscr{V}$ be a suitably chosen space of kinematically admissible displacement fields. For given boundary conditions (1) and for a specific (i.e., "frozen") elasticity tensor field $\boldsymbol{C}$, the displacement field $\mathbf{u}$ that satisfies the equilibrium equation is the (unique) element of the set $\mathscr{E}_C$ defined as follows:

$$\mathscr{E}_C = \left\{\mathbf{u}\in\mathscr{V}\,\middle|\,\int_{\Omega}\boldsymbol{\varepsilon}(\mathbf{u})\cdot\boldsymbol{C}\boldsymbol{\varepsilon}(\mathbf{v})\mathrm{d}v = W[\mathbf{v},\bar{\mathbf{b}}(\alpha)] \quad \forall\mathbf{v}\in\mathscr{V}\right\}. \tag{7}$$

The work W given by eqn (6) is viewed as a global measure of the stiffness of a structure and takes into account the prescribed domain Ω and boundary conditions. The stiffest structure corresponds to the one which minimizes W for fixed boundary conditions, eqn (1), when the elasticity tensor field $\boldsymbol{C}$ is taken as variable.

2.4. Optimization problem: maximum structural stiffness

The optimization problem is stated as: for a linearly elastic material occupying a given domain Ω and for boundary conditions given by eqn (1), find the minimizer

$$\boldsymbol{C} = \sum_{\mu=1}^{6}\alpha_\mu\mathbf{A}_\mu\otimes\mathbf{A}_\mu$$

of the following expression:

$$\min_{\{\alpha_\mu\}\in\mathscr{S}_{\alpha_\mu}}\ \min_{\{\mathbf{A}_\mu\}\in\mathscr{S}_{\mathbf{A}_\mu}} W[\mathbf{u},\bar{\mathbf{b}}(\alpha)] \tag{P1}$$

where $\mathbf{u}\in\mathscr{E}_c$ (hence the admissible field $\mathbf{u}$ is the equilibrium solution for a given field $\boldsymbol{C}$) and α is given by eqn (4). The two minimization parts reflect a decomposition of the optimization problem in terms of two sets of variables (eigenvalues $\{\alpha_\mu\}$ and eigentensors $\{\mathbf{A}_\mu\}$ of $\boldsymbol{C}$). The sets $\mathscr{S}_{\alpha_\mu}$ and $\mathscr{S}_{\mathbf{A}_\mu}$ correspond to the constraints on the optimization variables. For the solution $\mathbf{u}$ of the elastostatic problem (i.e., $\mathbf{u}\in\mathscr{E}_c$), the dependence of W on $\boldsymbol{C}$, as given by eqn (6), is two-fold: $\mathbf{u}$ depends implicitly on the constitutive law and, by assumption, the body force depends explicitly on the maximum eigenvalue of $\boldsymbol{C}$. Formally, W also depends on Ω and the prescribed boundary conditions, but these are considered fixed.

The problem (P1) can be simplified considerably by virtue of a saddle point theorem as proved in Bendsøe et al. (1994) (Jog et al., 1993; Lipton, 1994). This is an established result in the optimization of structures. Nonetheless, in order to illustrate the explicit role played by the eigenvalues and eigentensors of $\boldsymbol{C}$ introduced in eqn (2) and to show that the saddle point theorem remains unaffected by a structure-dependent body force of the form of eqn (5), it is useful to revisit this result. More significantly, the analysis presented here provides an interpretation of the solution of the optimization problem in terms of a positive definite material, as opposed to previous solutions which were given in terms of semi-definite materials. To this end, the admissible displacement fields in (P1) are characterized as minimizers of the potential energy (instead of enforcing the equilibrium equations). The potential energy is given by $\Pi = U - W$, where

$$U = \frac{1}{2}\int_{\Omega} \boldsymbol{\varepsilon} \cdot \boldsymbol{C}\boldsymbol{\varepsilon}\,\mathrm{d}v$$

is the total strain energy. The minimum value of the potential energy is $\Pi_* = -\frac{1}{2}W_*$, hence, reversing the sign of the objective functional, the problem (P1) can be expressed alternatively as: find the maximizing eigenvalue fields, eigentensor fields and minimizing displacement field of the following expression:

$$\max_{\{\alpha_\mu\}\in\mathscr{S}_{\alpha_\mu}} \max_{\{\mathbf{A}_\mu\}\in\mathscr{S}_{\mathbf{A}_\mu}} \min_{\mathbf{v}\in\mathscr{V}} \Pi\,[\alpha_\mu, \mathbf{A}_\mu; \mathbf{v}], \tag{P2}$$

where, since

$$\boldsymbol{\varepsilon} \cdot \boldsymbol{C}\boldsymbol{\varepsilon} = \sum_{\mu=1}^{6} \alpha_\mu (\boldsymbol{\varepsilon} \cdot \mathbf{A}_\mu)^2,$$

the potential energy is

$$\Pi[\alpha_\mu, \mathbf{A}_\mu; \mathbf{v}] = \sum_{\mu=1}^{6} \int_{\Omega} \tfrac{1}{2}\alpha_\mu (\boldsymbol{\varepsilon}(\mathbf{v}) \cdot \mathbf{A}_\mu)^2 \mathrm{d}v - W[\mathbf{v}, \bar{\mathbf{b}}(\alpha)]. \tag{8}$$

The minimization part in (P2) corresponds to the elastostatic problem for a given elasticity tensor field. In view of the saddle point theorem mentioned above, the two inner problems in (P2) can be interchanged (the theorem can be applied because it is assumed that $\bar{\mathbf{b}}$ is not a function of $\mathbf{A}_\mu$). It is worth noting that the outermost maximization problem cannot be interchanged with the innermost minimization problem since they provide different solutions. After interchanging the two inner problems in (P2), the innermost problem becomes, for given $\mathbf{v}$ and $\{\alpha_\mu\}_{\mu=1}^{6}$, a local algebraic problem, i.e., find the maximizers, at each point $\mathbf{x} \in \boldsymbol{\Omega}$, in the following expression:

$$\max_{\{\mathbf{A}_\mu\}\in\mathscr{S}_{\mathbf{A}_\mu}} \sum_{\mu=1}^{6} \tfrac{1}{2}\alpha_\mu (\boldsymbol{\varepsilon}(\mathbf{v}) \cdot \mathbf{A}_\mu)^2. \tag{9}$$

To provide a 'geometrical' interpretation of the problem notice that, since $\boldsymbol{\varepsilon} \cdot \boldsymbol{C}\boldsymbol{\varepsilon} = (\varepsilon \otimes \boldsymbol{\varepsilon}) \cdot \boldsymbol{C}$, then eqn (9) is equivalent to

$$\max_{\{\mathbf{A}_\mu \mathbf{A}_\mu\}} \left[\tfrac{1}{2}(\boldsymbol{\varepsilon}(\mathbf{v}) \otimes \boldsymbol{\varepsilon}(\mathbf{v})) \cdot \sum_{\mu=1}^{6} \alpha_\mu (\mathbf{A}_\mu \otimes A_\mu) \right], \tag{10}$$

where the maximization is carried out for six 4-tensors of the form $\mathbf{A}_\mu \otimes \mathbf{A}_\mu$ and the problem is linear since the goal is to maximize the scalar product of 'vectors' in a 4-tensor space, where both $\{\alpha_\mu\}_{\mu=1}^{6}$ and $\boldsymbol{\varepsilon} \otimes \boldsymbol{\varepsilon}$ are fixed. The objective function in the inner problem of eqn (10) (or eqn (9)) admits the following trivial bound:

$$\tfrac{1}{2}(\boldsymbol{\varepsilon}\otimes\boldsymbol{\varepsilon})\cdot\sum_{\mu=1}^{6}\alpha_\mu(\mathbf{A}_\mu\otimes\mathbf{A}_\mu)\leqslant\tfrac{1}{2}\max_{1\leqslant\mu\leqslant 6}\{\alpha_\mu\}\,\|\boldsymbol{\varepsilon}\otimes\boldsymbol{\varepsilon}\|=\tfrac{1}{2}\alpha\boldsymbol{\varepsilon}\cdot\boldsymbol{\varepsilon}. \tag{11}$$

This upper bound is independent of $\{\mathbf{A}_\mu\}$ and depends on the local values of $\{\alpha_\mu\}$ only via their maximum α. To reach this bound, one could choose the eigentensor corresponding to the maximum eigenvalue as

$$\mathbf{A}=\hat{\boldsymbol{\varepsilon}}=\frac{\boldsymbol{\varepsilon}}{\|\boldsymbol{\varepsilon}\|},$$

while the other eigentensors can be chosen arbitrarily as long as they form an orthonormal basis. This choice can be made regardless of the multiplicity of the maximum eigenvalue (i.e., one of the eigentensors corresponding to α can be defined as above). With this specific choice the upper bound is reached, thus providing an optimal solution for the set $\{\mathbf{A}_\mu\}$, valid for any point $\mathbf{x}\in\Omega$, in terms of the (kinematically admissible) displacement field $\mathbf{v}$. An explicit expression for the eigentensors different than $\hat{\boldsymbol{\varepsilon}}$ is not required for the present purposes. From eqns (8), (11) and the definition of $\mathscr{S}_{\alpha_\mu}$, one can see that if the upper bound is reached, then the problem (P2) depends only on the maximum eigenvalue α. In that case, the eigenvalues smaller than α can be chosen arbitrarily as long as they belong to $\mathscr{S}_{\alpha_\mu}$. This problem has no unique solution, however a convenient choice is to set these eigenvalues to the lower bound α_m in order to guarantee that α is the maximum. If the multiplicity of α is greater than one, then the above choice corresponds to reducing its multiplicity to one. The essential point is that any admissible choice provides the same final result (i.e., upper bound). The optimal material can now be expressed as

$$\boldsymbol{C}_0=\boldsymbol{C}_0(\alpha,\mathbf{v})=(\alpha-\alpha_m)\frac{\varepsilon(\mathbf{v})\otimes\varepsilon(\mathbf{v})}{\|\varepsilon(\mathbf{v})\|^2}+\alpha_m\boldsymbol{I}. \tag{12}$$

The coupling between the optimization and elastostatic problems is reflected in the fact that the elasticity tensor depends on the strain field. It is noted in passing that $\boldsymbol{C}_0$ is an orthotropic material with symmetry directions aligned (locally) with the principal directions of the strain tensor: to see this, suppose that $\mathbf{Q}$ is a proper orthogonal second order transformation that commutes with $\boldsymbol{\varepsilon}$, hence $\mathbf{Q}$ and $\boldsymbol{\varepsilon}$ have the same principal directions in an Euclidean space. Since $\boldsymbol{C}_0$ transforms as $\Sigma_{\mu=1}^{6}\ \alpha_\mu\mathbf{Q}\mathbf{A}_\mu\mathbf{Q}^T\otimes\mathbf{Q}\mathbf{A}_\mu\mathbf{Q}^T$ and since $\mathbf{Q}\boldsymbol{\varepsilon}\mathbf{Q}^T=\boldsymbol{\varepsilon}$ then it follows from (12) that $\mathbf{Q}$ belongs to the material symmetry group of $\boldsymbol{C}_0$. The orthotropy of the optimal material is, in fact, a well-known result (see, e.g., Pedersen (1989), Cowin (1994)). However, the specific form (12) has a new ingredient compared to the one proposed by Bendsoøe et al. (1994), i.e., the term $\alpha_m\boldsymbol{I}$, which provides the required stability of the material. Recently, Taylor (1998) proposed a general framework where, among other things, terms such as $\alpha_m\boldsymbol{I}$ can be easily introduced. The essential term in the material (12) is a 4-tensor perpendicular projection onto the 2-tensor space spanned by the strain tensor (which can be interpreted as an optimal "reinforcement," characterized by the maximum principal stiffness α). For illustration purposes, the material properties of the orthotropic material can also be expressed in terms of the (three) Young moduli E_i, the (three independent) Poisson's ratios v_{ij} and (three) shear moduli G_{ij}. Let $\hat{\varepsilon}_i$, $i=1,2,3$, be the principal values of the (unit) strain tensor $\hat{\boldsymbol{\varepsilon}}=\boldsymbol{\varepsilon}/\|\boldsymbol{\varepsilon}\|$; for an *arbitrary* strain tensor $\boldsymbol{\gamma}$, the corresponding stress is, from eqn

(12), $\boldsymbol{\sigma} = \boldsymbol{C}_{0\gamma} = (\alpha - \alpha_m)(\hat{\boldsymbol{\varepsilon}} \cdot \boldsymbol{\gamma})\hat{\boldsymbol{\varepsilon}} + \alpha_{m\gamma}$. Hence, the elastic moduli referred to a Cartesian basis aligned locally with the principal directions of $\boldsymbol{\varepsilon}(\mathbf{v})$ are given by

$$E_i = \frac{\alpha\alpha_m}{\alpha_m + (1 - \hat{\varepsilon}_i^2)\alpha}, \quad \nu_{ij} = \frac{(\alpha - \alpha_m)\hat{\varepsilon}_i\hat{\varepsilon}_j}{\alpha_m + (1 - \hat{\varepsilon}_i^2)\alpha}, \quad G_{ij} = \tfrac{1}{2}\alpha_m.$$

Moreover, the stress field associated with the optimal strain field $\boldsymbol{\varepsilon}(\mathbf{v})$ takes the following simple form:

$$\boldsymbol{\sigma} = \boldsymbol{C}_0\boldsymbol{\varepsilon} = \alpha\varepsilon. \tag{13}$$

By optimal strain field it is meant that $\boldsymbol{\varepsilon}$ is the strain field in the elastostatic problem after the material $\boldsymbol{C}$ has been optimized analytically with respect to the set $\{\mathbf{A}_\mu\}$. Although the elasticity tensor given by eqn (12) is anisotropic, its restriction to the optimal strain field, given by eqn (13), corresponds formally to an isotropic material with material coefficients $3\kappa = 2\mu = \alpha$ or, equivalently, to $\nu = 0$ and $E = \alpha$ where ν is Poisson's ratio and E is Young's modulus. This formal interpretation is valid since the optimization and elastostatic problems are being solved simultaneously, hence the optimal material properties are coupled to the optimal displacement field (this is referred to as a self-adaptive material, although it is a mathematical rather than a physical property).

In view of eqn (12), the problem (P1) can be simplified in terms of the scalar field α (maximum principal stiffness), i.e., find the minimizer α of

$$\min_{\alpha \in \mathscr{S}_\alpha} W[\mathbf{u}, \bar{\mathbf{b}}(\alpha)], \tag{P3}$$

where $\mathbf{u} \in \mathscr{E}_\alpha$,

$$\mathscr{E}_\alpha = \left\{ \mathbf{u} \in \mathscr{V} \,\middle|\, \int_\Omega \alpha\varepsilon(\mathbf{u}) \cdot \varepsilon(\mathbf{v})\mathrm{d}v = W[\mathbf{v}, \bar{\mathbf{b}}(\alpha)]\ \forall \mathbf{v} \in \mathscr{V} \right\} \tag{14}$$

and

$$\mathscr{S}_\alpha = \left\{ \alpha \,\middle|\, \alpha_m \leqslant \alpha(\mathbf{x}) \leqslant \alpha_M, \int_\Omega \alpha \mathrm{d}v \leqslant R, \forall \mathbf{x} \in \Omega \right\}.$$

At points where the maximum principal stiffness α is equal to its lower bound α_m, the interpretation is that no reinforcement is required. The formulation (P3) provides a significant simplification from (P1) since the minimization is carried out for a single scalar field as opposed to a tensor field. For given α, the local form of the elastostatic problem of the optimal strain $\boldsymbol{\varepsilon}$ at points where α and $\boldsymbol{\varepsilon}$ are smooth is, from eqn (14)

$$\mathrm{div}(\alpha\boldsymbol{\varepsilon}) + \bar{\mathbf{b}}(\alpha) = 0 \quad \text{in } \Omega, \tag{15a}$$

$$\boldsymbol{\sigma}\mathbf{n} = \alpha\boldsymbol{\varepsilon}\mathbf{n} = \hat{\mathbf{t}} \quad \text{on } \partial\Omega_t, \tag{15b}$$

$$\mathbf{u} = \mathbf{0} \quad \text{on } \partial\Omega_u, \tag{15c}$$

where $\mathbf{n}$ is the outward unit normal vector to the boundary $\partial\Omega$. Observe that for the shape optimization, 'holes' are identified via a limit process when $\alpha_m/\alpha_M \to 0$, for which uniqueness is preserved. However, in the limit, the material becomes semi-definite, which is in fact one of the limitations of the 'true' single-loading shape optimization problem. Hence, after the solution of (P3) has been identified (optimal structure) and is considered to be fixed, α_m needs to be bounded away from zero by an amount comparable to possible perturbations on the surface loads in order to have a 'robust' structure.

3. Optimality conditions

The optimality conditions correspond to a set of necessary relations satisfied locally by the minimizer of (P3) and can be used for numerical methods or to solve some simple problems in closed form as will be shown in Section 4. To obtain these conditions, consider the Lagrangian L given by

$$L[\alpha;\lambda_m,\lambda_M,\Lambda] = W_0[\mathbf{u},\bar{\mathbf{b}}(\alpha)] + \int_\Omega [\lambda_m(\alpha_m-\alpha)-\lambda_M(\alpha-\alpha_M)]\mathrm{d}v + \Lambda\left\{\int_\Omega \alpha \mathrm{d}v - R\right\}, \tag{16}$$

where $\lambda_m = \lambda_m(\mathbf{x})$, $\lambda_M = \lambda_M(\mathbf{x})$ are Lagrange multipliers associated with the upper and lower bound constraints respectively and Λ is the (constant) multiplier corresponding to the resource constraint. As mentioned before, the displacement field $\mathbf{u}$ is viewed as a function of α since it can be obtained from eqn (14). Hence, in terms of the Lagrangian L, the problem (P3) can be expressed as: find the maximizers λ_m, λ_M, Λ and minimizer α in the following expression:

$$\max_{\lambda_m\geqslant 0,\lambda_M\geqslant 0,\Lambda\geqslant 0}\ \min_\alpha L[\alpha;\lambda_m,\lambda_M,\Lambda].$$

The gradient of L with respect to α can be computed as follows: consider a variation $\delta\alpha$ which induces variations $\delta\mathbf{u}$ and $\delta\bar{\mathbf{b}}$. Let L' be the gradient of L, therefore, from eqns (5), (6), (12) and (16) it follows that

$$\delta L[\delta\alpha] = \int_\Omega L'\delta\alpha \mathrm{d}v = \int_\Omega (\delta\bar{\mathbf{b}}\cdot\mathbf{u}+\bar{\mathbf{b}}\cdot\delta\mathbf{u})\mathrm{d}v + \int_{\partial\Omega_t}\bar{\mathbf{t}}\cdot\delta\mathbf{u}\mathrm{d}a - \int_\Omega(\lambda_m-\lambda_M-\Lambda)\delta\alpha\mathrm{d}v. \tag{17}$$

To complete the calculation, one can take variations in eqn (14), then integrate by parts, choose $\mathbf{v}=\mathbf{u}$, and use eqns (15) and (17) to get

$$\int_\Omega L'\delta\alpha\mathrm{d}v = \int_\Omega 2\delta\bar{\mathbf{b}}\cdot\mathbf{u}\mathrm{d}v - \int_\Omega \delta\alpha(\boldsymbol{\varepsilon}(\mathbf{u})\cdot\boldsymbol{\varepsilon}(\mathbf{u})+\lambda_m-\lambda_M-\Lambda)\mathrm{d}v.$$

Since $\delta\bar{\mathbf{b}} = \bar{\mathbf{b}}'\delta\alpha$, where $\bar{\mathbf{b}}'$ is the gradient of $\bar{\mathbf{b}}$ with respect to α, then, assuming enough differentiability, the local form of the gradient is

$$L' = 2\bar{\mathbf{b}}\cdot\mathbf{u}-\boldsymbol{\varepsilon}(\mathbf{u})\cdot\boldsymbol{\varepsilon}(\mathbf{u})-\lambda_m+\lambda_M+\Lambda. \tag{18}$$

Therefore, the optimality conditions (Karush–Kuhn–Tucker conditions) are, $\forall\mathbf{x}\in\Omega$,

$$\begin{aligned} &L' = 0, \\ &\lambda_m(\alpha_m - \alpha) = 0, \qquad \lambda_m \geqslant 0, \\ &\lambda_M(\alpha - \alpha_M) = 0, \qquad \lambda_M \geqslant 0, \\ &\Lambda\left\{\int_\Omega \alpha \mathrm{d}v - R\right\} = 0, \quad \Lambda \geqslant 0. \end{aligned} \tag{19}$$

To interpret the optimality conditions one can use the following domains:

$$\begin{aligned} \Omega_m &= \{\mathbf{x} \in \Omega | \alpha(\mathbf{x}) = \alpha_m\}, \\ \Omega_M &= \{\mathbf{x} \in \Omega | \alpha(\mathbf{x}) = \alpha_M\}, \\ \Omega_i &= \{\mathbf{x} \in \Omega | \alpha_m < \alpha(\mathbf{x}) < \alpha_M\}. \end{aligned} \tag{20}$$

Hence, since $\lambda_m = 0$ for $\mathbf{x} \in \Omega_M$, $\lambda_M = 0$ for $\mathbf{x} \in \Omega_m$, and $\lambda_m = \lambda_M = 0$ for $\mathbf{x} \in \Omega_i$, the optimality conditions of eqn (19) become

$$\begin{aligned} &\Lambda = \boldsymbol{\varepsilon}(\mathbf{u}) \cdot \boldsymbol{\varepsilon}(\mathbf{u}) - 2\bar{\mathbf{b}}' \cdot \mathbf{u}, && && \mathbf{x} \in \Omega_i, \\ &\Lambda - \lambda_m = \boldsymbol{\varepsilon}(\mathbf{u}) \cdot \boldsymbol{\varepsilon}(\mathbf{u}) - 2\bar{\mathbf{b}}' \cdot \mathbf{u}, && \lambda_m > 0, && \mathbf{x} \in \mathring{\Omega}_m, \\ &\Lambda + \lambda_M = \boldsymbol{\varepsilon}(\mathbf{u}) \cdot \boldsymbol{\varepsilon}(\mathbf{u}) - 2\bar{\mathbf{b}}' \cdot \mathbf{u}, && \lambda_M > 0, && \mathbf{x} \in \mathring{\Omega}_M, \\ &\Lambda\left\{\int_\Omega \alpha \mathrm{d}v - R\right\} = 0, && \Lambda \geqslant 0, && \mathbf{x} \in \Omega. \end{aligned} \tag{21}$$

If $\bar{\mathbf{b}} = \bar{\mathbf{b}}' = 0$ then one can prove that $\Lambda > 0$ and hence that the resource constraint is active (i.e., satisfied as an equality; Bendsøe et al., 1994). Clearly, to have a non-trivial solution in the case when the resource constraint is active, the upper bound R in the resource constraint has to satisfy $\alpha_m|\Omega| < R < \alpha_M|\Omega|$. However, if $\bar{\mathbf{b}}' \not\equiv 0$, it is possible that $\Lambda = 0$ and the problem (P3) has a non-trivial solution. This case will be illustrated by an example in Section 4. Furthermore, recall that it was assumed that $\bar{\mathbf{b}}$ is a monotonically increasing function of α hence, in view of eqn (18), if $\bar{\mathbf{b}} \cdot \mathbf{u} > 0$, the local effect of the term $2\bar{\mathbf{b}}' \cdot \mathbf{u}$ is to increase the value of the gradient of the Lagrangian L. Conversely, if $\bar{\mathbf{b}} \cdot \mathbf{u} < 0$ (hence $\bar{\mathbf{b}}' \cdot \mathbf{u} < 0$), the local effect of the term $2\bar{\mathbf{b}}' \cdot \mathbf{u}$ is to decrease L'. Therefore, compared to the case when the body force is zero, the optimal structure is locally weaker (smaller maximum principal stiffness α) if $\bar{\mathbf{b}} \cdot \mathbf{u} > 0$ and locally stiffer (greater α) otherwise. This effect will be illustrated by a three-dimensional example in the next section.

4. Example: the optimal rotating structure

As an example of a structure-dependent body force, consider the problem of finding the stiffest structure occupying a domain Ω and rotating at a constant angular velocity ω. Let $\{\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z\}$ be a cylindrical orthonormal basis where $\mathbf{e}_z$ is aligned with the axis of rotation and $\mathbf{e}_r$, $\mathbf{e}_\theta$ are fixed with respect to Ω ($\mathbf{e}_r$ points in the outward radial direction). For illustration purposes, suppose that the optimization is carried out with a material for which the mass density ρ is related to the maximum principal stiffness α linearly, i.e.,

$$\rho = \kappa\alpha,$$

where κ is a positive constant. It is worth noting that for nonlinear functions ρ the corresponding problem is nonlinear and might not have a solution. The inertial force can be modeled in the elastostatic problem as the body force

$$\bar{\mathbf{b}}(\alpha) = \omega^2 r\rho\mathbf{e}_r = \omega^2 r\kappa\alpha\mathbf{e}_r, \tag{22}$$

where r is the orthogonal distance from a given point in Ω to the axis of rotation. To obtain some insight in the problem, it is possible to solve analytically a one-dimensional example. To this end, consider a prismatic region Ω of length l and cross-sectional area a. Suppose that the displacement field and the maximum principal stiffness α depend on r only, $r \in [0, l]$. As an ansatz, assume that the domains defined by eqn (20) are such that $\Omega_M = [0, r_1]$, $\Omega_i = [r_1, r_2]$, $\Omega_m = [r_2, l]$ and that the resource constraint is not active (i.e., $\Lambda = 0$). It will be shown a posteriori that these assumptions are appropriate to characterize solutions with boundary conditions $u(0) = 0$ and $\sigma = \alpha(l)u'(l) = \sigma_0 > 0$, where $u(r)$ is the radial displacement. The optimality conditions of eqn (21) become

$$\lambda_M(r) = (u'(r))^2 - 2\kappa\omega^2 ru(r), \quad \lambda_M(r) \geqslant 0, \quad 0 \leqslant r \leqslant r_1 \tag{23a}$$

$$(u'(r))^2 - 2\kappa\omega^2 ru(r) = 0 \quad r_1 \leqslant r \leqslant r_2 \tag{23b}$$

$$\lambda_m(r) = (u'(r))^2 + 2\kappa\omega^2 ru(r), \quad \lambda_m(r) \geqslant 0, \quad r_2 \leqslant r \leqslant l \tag{23c}$$

and the balance of linear momentum is, from eqn (15a),

$$(\alpha u')' + \kappa\omega^2 r\alpha = 0, \quad 0 \leqslant r \leqslant l. \tag{24}$$

Observe that, since $\alpha = \alpha_M > 0$ in Ω_M and $\alpha = \alpha_m > 0$ in Ω_m, then eqn (24) becomes

$$u''(r) + \kappa\omega^2 r = 0, \quad \forall r \in \Omega_M \bigcup \Omega_m.$$

The displacement field can be determined in Ω_M and Ω_m up to a constant (say, c_M and c_m) by solving eqn (24) and using the boundary conditions. This, in turn, provides an expression for $\lambda_M(r)$ and $\lambda_m(r)$ from eqn (23a,c). Furthermore, solving eqn (23b) gives the displacement field in Ω_i up to a constant (say, c_1). With this displacement field one can solve eqn (24) in Ω_i and determine $\alpha(r)$ up a constant (say, c_2). The solution is displayed more conveniently in nondimensional form. Define

$$\bar{u} = \frac{u}{l_\gamma}, \quad \bar{r} = \frac{r}{l}, \quad \bar{\alpha} = \frac{\alpha}{\alpha_M}, \quad \text{where } \gamma = \kappa\omega^2 l^2,$$

and the following nondimensional parameters:

$$f = \frac{2\sigma_0}{\alpha_M \kappa\omega^2 l^2}, \quad \beta = \frac{\alpha_m}{\alpha_M}. \tag{25}$$

The parameter f, referred to as the loading parameter, includes the applied loads, the upper bound of the material properties and a term related to the body force, so it can be interpreted as the ratio of surface to body forces. The parameter β, referred to as the moduli parameter, corresponds to the nondimensional ratio of the lower and upper bounds of the reinforcement or, equivalently, to the nondimensional modulus of the term $\alpha_m \boldsymbol{I}$ of the optimal material $\boldsymbol{C}_0$. The displacement field and the maximum principal stiffness (reinforcement) are given by

$$\bar{u}(\bar{r}) = \begin{cases} \frac{1}{6}\bar{r}(c_M - \bar{r}^2) & \bar{r} \in \Omega_M = [0, \bar{r}_1], \\ \frac{1}{18}(3c_1 + 2\bar{r}^{3/2})^2 & \bar{r} \in \Omega_i = [\bar{r}_1, \bar{r}_2], \\ \frac{1}{6}[c_m + \bar{r}(3c_0^2 - \bar{r}^2)] & \bar{r} \in \Omega_m = [\bar{r}_2, 1], \end{cases} \tag{26}$$

and

$$\bar{\alpha}(\bar{r}) = \begin{cases} 1 & \bar{r} \in \Omega_M = [0, \bar{r}_1], \\ \dfrac{c_2}{\sqrt{\bar{r}}(3c_1 + 2\bar{r}^{3/2})^2} & \bar{r} \in \Omega_i = [\bar{r}_1, \bar{r}_2], \\ \beta & \bar{r} \in \Omega_m = [\bar{r}_2\ 1], \end{cases} \tag{27}$$

where the constant c_0 in (26) is $c_0 = \sqrt{1 + f/\beta}$. Two relevant nondimensional quantities can be identified in this solution: the loading and moduli parameters f and β defined in eqn (25) which can be used to characterize the different loading cases. The constants c_m, c_M, c_1, c_2, $\bar{r}_1$, and $\bar{r}_2$ are determined as follows: since $\bar{r}_1$ and $\bar{r}_2$ represent the location of the boundaries between Ω_M, Ω_i, and Ω_i, Ω respectively, then, for an admissible solution, it is required that $\lambda_M(\bar{r}_1) = \lambda_m(\bar{r}_2) = 0$. These conditions can be satisfied by choosing c_M and c_M as follows:

$$c_M = (9 + 2\sqrt{15})\bar{r}_1^2, \quad c_m = \frac{1}{4\bar{r}_2}(3c_0^4 - 18c_0^2\bar{r}_2^2 + 7\bar{r}_2^4).$$

Furthermore, two additional relations can be obtained from the continuity of $\bar{u}$ at $\bar{r} = \bar{r}_1$ and $\bar{r} = \bar{r}_2$, i.e.,

$$c_1 = \frac{1}{3}\left(1 + \sqrt{15}\right)\bar{r}_1^{3/2} = \frac{1}{6\sqrt{\bar{r}_2}}\left(3c_0^2 - 7\bar{r}_2^2\right).$$

There are other possible values for c_M and c_1 (c_m is unique), but these can be discarded a posteriori based on the requirements that the solution should not be singular and that the multipliers λ_m and λ_M should be positive in [0, $\bar{r}_1$) and ($\bar{r}_2$, 1] respectively. Finally, the value of c_2 and an additional relation between $\bar{r}_1$ and $\bar{r}_2$ can be obtained by enforcing the continuity of $\bar{\alpha}$ at $\bar{r} = \bar{r}_1$ and $\bar{r} = \bar{r}_2$ which gives

$$c_2 = 6(4 + \sqrt{15})\bar{r}_1^{7/2} = \beta\sqrt{\bar{r}_2}\{(1 + \sqrt{15})\bar{r}_1^{3/2} - 2\bar{r}_2^{3/2}\}^2.$$

The values of $\bar{r}_1$ and $\bar{r}_2$ can be computed numerically from the above relations. As f increases, so do $\bar{r}_1$ and $\bar{r}_2$. For some values of f, $\bar{r}_2 \geqslant 1$, then the solution needs to be modified since $\Omega_m = \varnothing$ (i.e., the solution never reaches its lower bound β). In that case the solution depends on the loading parameter f but not on β. The value of $\bar{r}_1$ can be obtained from the boundary condition $\alpha(l)u'(l) = \sigma_0$, which can be expressed as

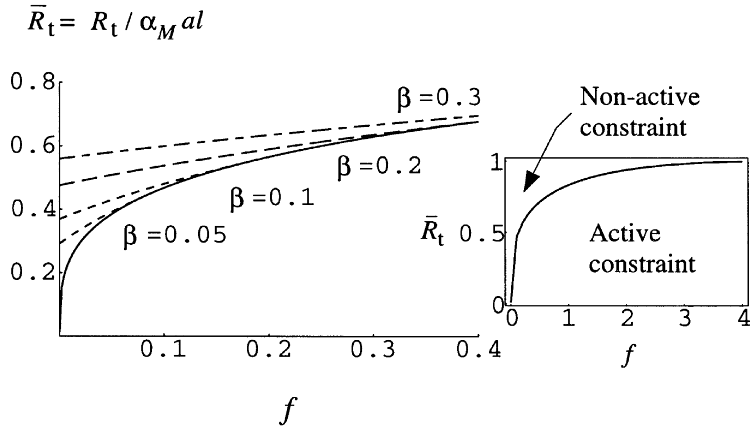

Fig. 1. One-dimensional case: normalized total resource $\bar{R}_t = R_t/(\alpha_M al)$ for different values of the loading and moduli parameters f and β. The insert corresponds to the limit case $\beta \to 0$ and shows the full range $0 \leqslant \bar{R}_t \leqslant 1$. For prescribed values of f and β, the resource constraint is not active if the prescribed $\bar{R}$ is above the corresponding $\bar{R}_t$ and active otherwise.

$$4(4+\sqrt{15})\bar{r}_1^{7/2} = f(2+(1+\sqrt{15})\bar{r}_1^{3/2}). \tag{28}$$

Finally, for f large enough, $\bar{r}_1 \geqslant 1$, hence the solution is trivial: $\bar{\alpha} \equiv 1$ and the boundary condition is satisfied with $c_M = 3(f+1)$. From eqn (27) one can obtain the total resource,

$$R_t = a\int_0^l \alpha dr,$$

where a is the cross-sectional area. Since $\bar{\alpha}(\bar{r})$ is a monotonically decreasing function from 1 to β as $\bar{r}$ ranges from 0 to 1, then $0 < \beta < \bar{R}_t < 1$ where $\bar{R}_t = R_t/(\alpha_M al)$. Therefore, one can always choose an upper bound $\bar{R} = R/(\alpha_M al)$ on the resource such that $\bar{R}_t < \bar{R} < 1$, i.e., such that the resource constraint is not active but $\bar{R}_t \neq 0$. This confirms the assumption that there are non-trivial solutions with $\Lambda = 0$. Clearly, if the prescribed $\bar{R}$ is such that $\bar{R}_t > \bar{R}$, then eqns (26)–(27) would not be an admissible solution since the resource constraint would be violated. However, an analytical expression for the solution when the resource constraint is active is not available.

Figure 1 represents the total resource $\bar{R}_t$ for different values of the moduli parameter β and loading parameter f. Observe that for values of f large enough, the solution does not depend on β anymore and all curves merge into a common envelope which can be thought of as the limit case $\beta \to 0$ for all values of f (see insert in Fig. 1). If, for a given pair β, f, the prescribed upper bound on the resource constraint $\bar{R}$ is above the corresponding curve $\bar{R}_t$, then the resource constraint is not active. The maximum principal stiffness $\bar{\alpha}(\bar{r})$ is shown in Fig. 2. for different values of β and a common loading parameter $f = 0.1$. The solid curve represents the case for which the solution does not depend on β for the given f (i.e., β is below some critical value β_*). In the first part of each curve, $\bar{\alpha}$ is at its upper bound then it decreases monotonically to its lower bound β (except for $\beta < \beta_*$). The corresponding minimum values of the objective functional are $\bar{W}_*(\beta = 0.3) = 5.11\times10^{-2}$, $\bar{W}_*(\beta = 0.2) = 4.07\times10^{-2}$, and $\bar{W}_* = 3.30\times10^{-2}$ for $\beta < \beta_*$. Although β is viewed

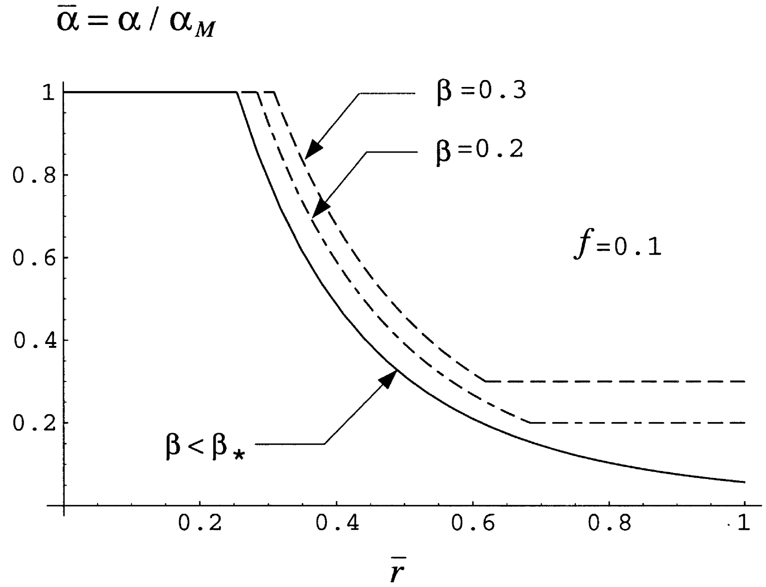

Fig. 2. One-dimensional case: value of the maximum principal stiffness $\bar{\alpha}$ (reinforcement) in the radial direction for various values of the moduli parameter β and a common loading parameter $f = 0.1$. The solid curve represents the limit case for which the solution does not depend on β.

as a fixed parameter, the optimal solution is reached for values of $\beta < \beta_*$. However, as mentioned before, β needs to be bounded away from zero in order to have a stable structure. Figure 3 corresponds to the displacement field as a function of $\bar{r}$ for different values of β. Observe that at $\bar{r} = 1$ the displacement of the solution that does not depend on β (solid line) is greater than for the other curves, however, the energy norm which measures structural stiffness is smaller. Similarly, as shown in Fig. 4, the stress distribution is also smaller for $\beta < \beta_*$, which corresponds to the stress-based notion of optimal structure. Figure 5 shows, for different values of β, the (negative) unconstrained gradient of the Lagrangian L as defined in eqn (16) (i.e., $-(L'+\lambda_m-\lambda_M)/\gamma^2 = \dot{\bar{u}}^2-2\bar{r}\bar{u}$, where the dot represents differentiation with respect to $\bar{r}$). For each curve, the value of the function in the first interval $\Omega_M = [0, \bar{r}_1)$ (where the local upper bound is active ($\bar{\alpha} = 1$) and where $L' = \lambda_M = 0$) corresponds to $\lambda_M/\gamma^2 > 0$; in the second interval $\Omega_i = [\bar{r}_1, \bar{r}_2]$ (where no local bounds are active) to $L' = \lambda_M = \lambda_m = 0$ and in the third interval $\Omega_m = (\bar{r}_2, 1]$ (where the local lower bound is active ($\bar{\alpha} = \beta$) and where $L' = \lambda_m = 0$) to $-\lambda_m/\gamma^2 < 0$. Observe that for $\beta < \beta_*$ (solid curve), there is no third interval since $\Omega_m = \emptyset$.

To illustrate that this procedure can be used in a more general setting, a three-dimensional example was solved using a finite element program. Consider again the case of the optimal distribution of material properties in order to obtain maximum structural stiffness of a rotating prismatic structure of length l and square cross-sectional area h^2. The side attached to the axis of rotation is modeled as a clamped end. The remaining sides of the structure are subjected to the following loads:

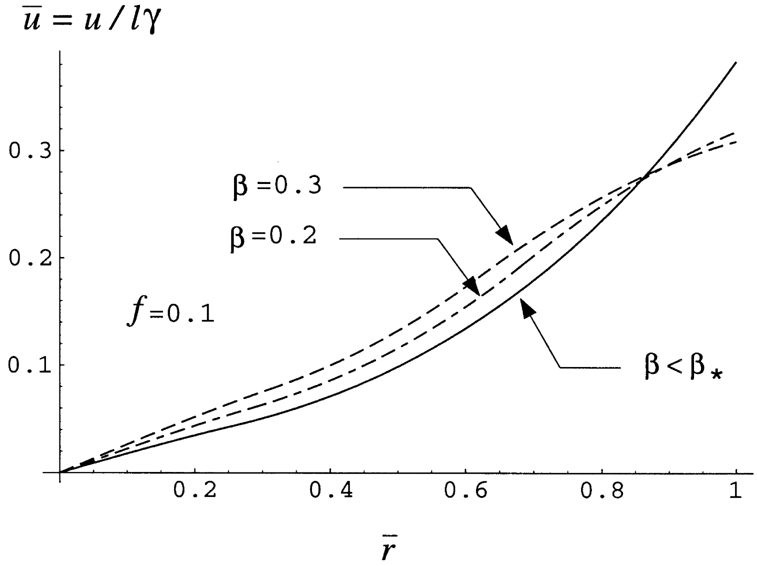

Fig. 3. One-dimensional case: radial displacement $\bar{u}(\bar{r})$ for various values of the moduli parameter β and a common loading parameter $f = 0.1$. The solid curve represents the limit case for which the solution does not depend on β.

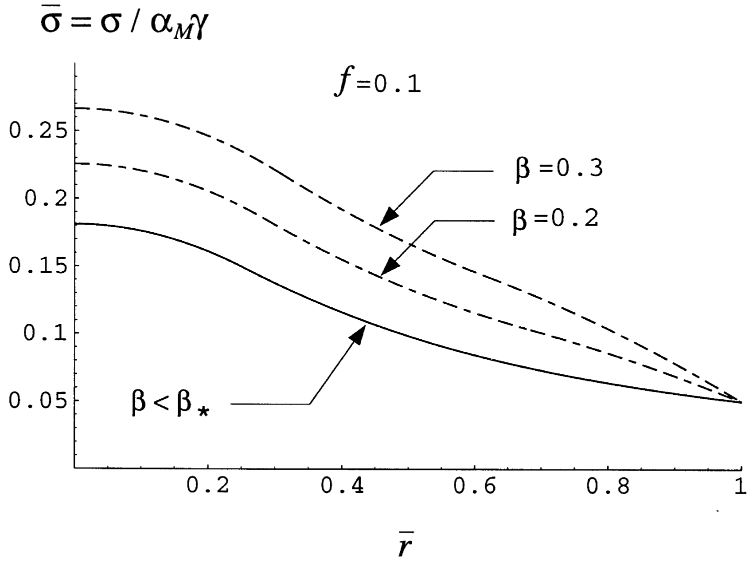

Fig. 4. One-dimensional case: stress distribution as a function of radial position for various values of the moduli parameter β and a common loading parameter $f = 0.1$ (i.e., the boundary condition is $\bar{\sigma}(1) = f/2$).

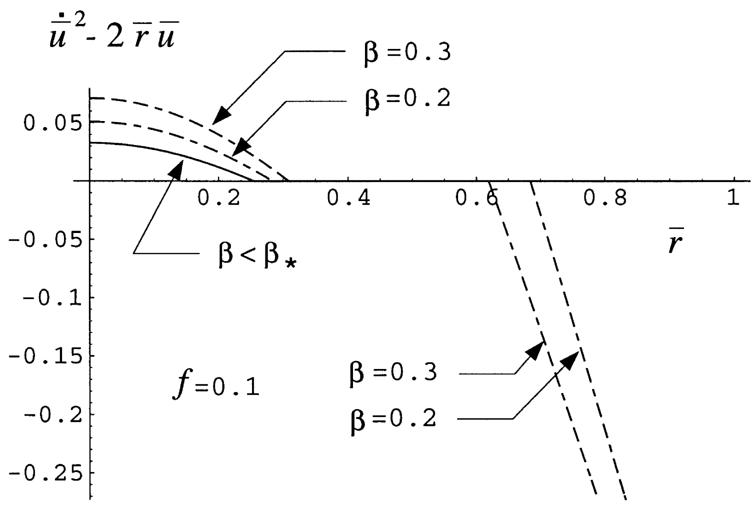

Fig. 5. One-dimensional case: (negative) unconstrained gradient of the Lagrangian ($= \dot{\bar{u}}^2 - 2\bar{r}\bar{u}$) for various values of the moduli parameter β and a common loading parameter $f = 0.1$. The solid curve represents the limit case for which the solution does not depend on β. See text for detailed explanation.

Case 1: shearing load:

$$\hat{\mathbf{t}} = \begin{cases} \tau_0 \mathbf{e}_z & \text{for } r = l \text{ (uniform shear stress)} \\ \mathbf{0} & \text{otherwise} \end{cases}$$

Case 2: torsional load:

$$\hat{\mathbf{t}} = \begin{cases} -\tau_0 \mathbf{e}_z & \text{for } r = l, \quad x \in [-h, -h+\varepsilon], \\ \tau_0 \mathbf{e}_z & \text{for } r = l, \quad x \in [h-\varepsilon, h], \\ \mathbf{0} & \text{otherwise} \end{cases}$$

where ε is a small number, the origin of coordinates is in the mid-section and x is measured in the $\mathbf{e}_\theta$ direction. The numerical values used here, for illustration purposes, are as follows: $l = 1$ m, $h = 0.1$ m, $\alpha_M = 10^{10}$ Pa, $\alpha_m = 10^8$ Pa, $\kappa = 10^{-7}$ s^2 m^{-2} and $\omega = 500$ RPM. In Case 1, $\tau_0 = 10^6$ Pa and in Case 2, $\tau_0 = 1.5 \times 10^6$ Pa, $\varepsilon = 0.01$ m. In both cases the resource constraint is active with $R = 0.5\, lh^2\alpha_M = 0.5 \times 10^8$ Pa m^3. The maximum principal stiffness $\bar{\alpha} = \alpha/\alpha_M$ is shown for the loading Case 1 (shearing load) in Fig. 6 (without a body force) and 7 (with a body force). Figure 8 represents contour plots of $\bar{\alpha}$ at the cross-section $\bar{r} = 0.75$ for the load Case 1; the section on the left corresponds to the case when the body force is absent and the right one when it is included. Observe that, in the latter cross-section, the top of the structure is in compression, thus $\bar{\mathbf{b}} \cdot \mathbf{u} < 0$, which results in a greater reinforcement than the bottom which is in tension (see end of Section 3). In contrast, when the body force is absent, the solution is symmetric with respect to the r, θ mid-plane, as shown in the left cross-section in Fig. 8. (compare also Figs 6 and 7). The heuristic interpretation is as follows: first, observe that in this example the body force always creates tension

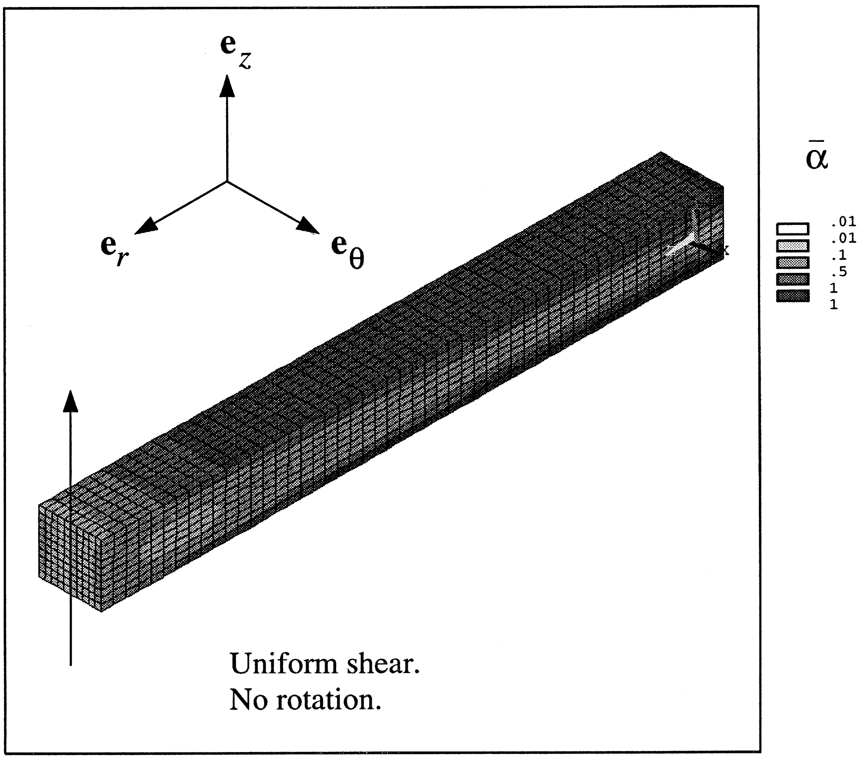

Fig. 6. Loading Case 1 (shearing load): contour plot of the maximum principal stiffness (reinforcement) without a body force. Observe the symmetry of the solution with respect to the $\mathbf{e}_r$, $\mathbf{e}_\theta$ mid-plane.

at any point of the structure. Now, consider a point of the structure which is in compression because of the applied shear load. If the material is locally stiffer (hence, the local mass density is higher), then the body force that tends to oppose compression would be higher resulting in an overall smaller displacement. The opposite effect occurs at points which, because of the applied shear load, are in tension.

For the loading Case 2 (torsional load), Fig. 9 shows the maximum principal stiffness when the body force is not present and Fig. 10 when it is. In Fig. 11, two cross-sections at $\bar{r} = 0.75$ are shown; the left one corresponds to the case without the body force and the right one when the body force is taken into account. The procedure reproduces the well-known optimal ‘circular’ layout (within the limitations of the mesh and the effect of the boundary conditions at $r = 0$ and $r = l$). The center of the cross-section is at the lower bound (hence, it can be interpreted as a ‘weak’ region, i.e., as a ‘hole’). This illustrates the ability of the method to obtain the optimal shape as a by-product of the material optimization. The effect of the body force is reflected in the gradual decrease of reinforcement from $r = 0$ to $r = l$, as shown by the cross-sections in Fig. 11 (compare also Figs 9 and 10).

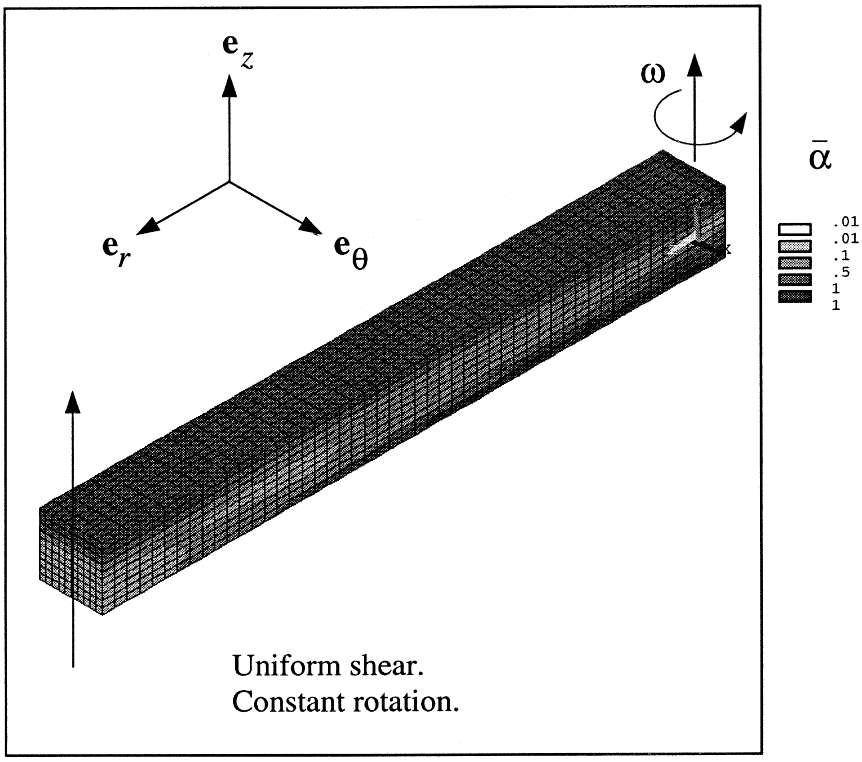

Fig. 7. Loading Case 1 (shearing load): contour plot of the maximum principal stiffness (reinforcement) with a body force. The upper section has a greater reinforcement than the lower section.

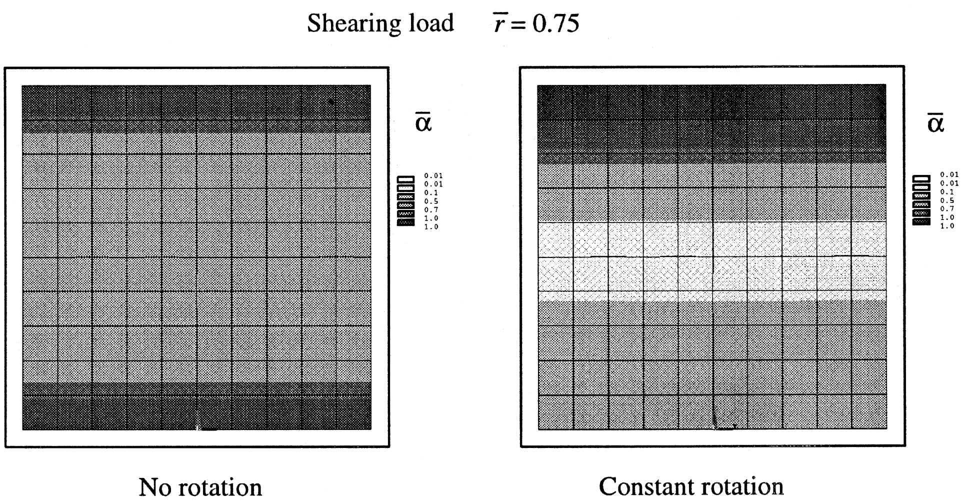

Fig. 8. Loading Case 1 (shearing load): cross-sectional contour plot of the maximum principal stiffness (reinforcement) at $\bar{r} = 0.75$; the left plot corresponds to the case without a body force (symmetric solution) and the right one to the case with body force (greater reinforcement in upper section which is in compression, lesser reinforcement in lower section which is in tension).

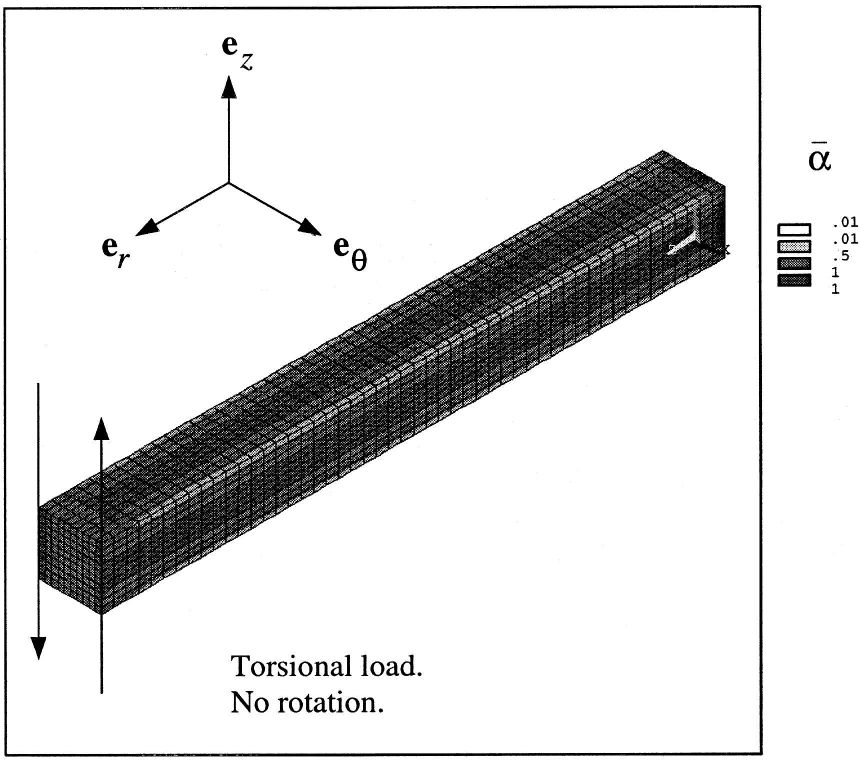

Fig. 9. Loading Case 2 (torsional load): contour plot of the maximum principal stiffness (reinforced material) without a body force. Observe that the distribution of reinforcement is prismatic, except close to $\bar{r} = 0$ and $\bar{r} = 1$ due to the boundary conditions.

5. Discussion and conclusions

The method presented here identifies a positive-definite optimal material that maximizes the structural stiffness when a structure-dependent body force is taken into account. Two interesting aspects arise due to the presence of the body force: as opposed to problems with zero body force, the resource constraint might not be active yet it is possible to have a non-trivial solution where

$$\int_{\Omega} \varphi(\alpha)\mathrm{d}v = R_* \neq 0.$$

This suggests that there is a 'natural' upper limit R_* on the amount of reinforcement material that needs to be used (i.e., trying to specify from the onset a greater amount will not affect the optimal solution). Also, it is not necessarily true that the presence of the body force will result in a local increase of the material properties in order to stiffen the structure. In fact, as shown by eqn (21), this depends on whether the scalar product of the body force and the displacement is negative (local stiffening) or positive (local weakening). It is usually assumed that at points on the surface where the loads are applied, the optimal layout would consist of some type of reinforcing structure

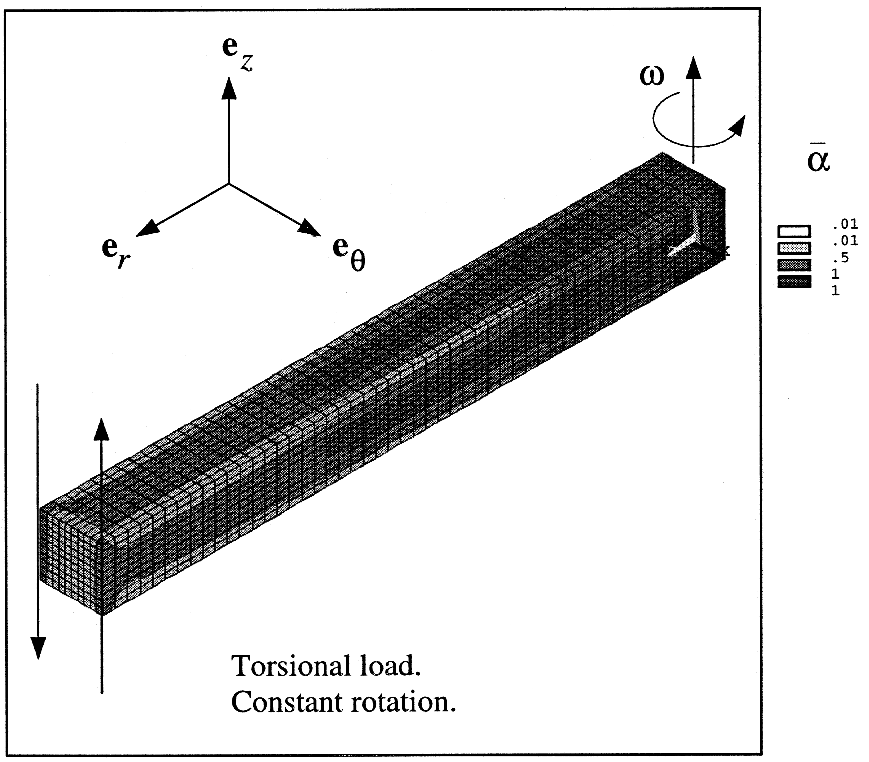

Fig. 10. Loading Case 2 (torsional load): contour plot of the maximum principal stiffness (reinforced material) with a body force. Observe the gradual decrease in reinforcement from $\bar{r} = 0$ to $\bar{r} = 1$.

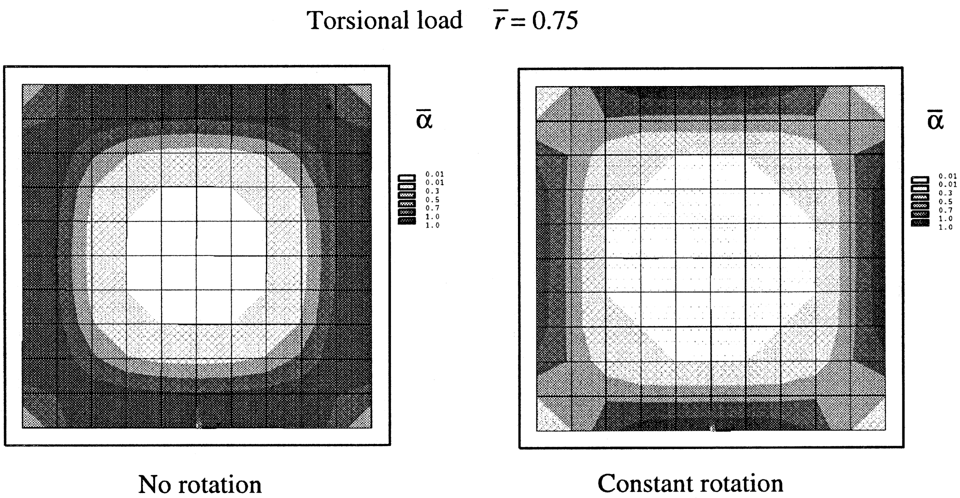

Fig. 11. Loading Case 2 (torsional load): cross-sectional contour plot of the maximum principal stiffness (reinforcement) at $\bar{r} = 0.75$; the left plot corresponds to the case without a body force and the right one to the case with body force.

to support those loads. However, this is not the case if the structure itself turns out to increase substantially the work done by the body force. Clearly, as shown by the optimality conditions, there is a compromise between these two opposite effects and the outcome can be quantified based on a characteristic magnitude of the surface loads, the upper bound on material properties, a characteristic magnitude of the body force and whether the resource constraint is active or not. If the inertial forces are small compared to the applied loads (i.e., large loading parameter f) then, as expected, the procedure converges to a solution similar to the case without body force. However, if the inertial forces are large (small f), then the procedure might fail to develop a suitable structure to support the loads and large displacement gradients could occur at those points. Nevertheless, a layout which includes a supporting structure for the loads can always be obtained by selecting an appropriately large value for the lower bound α_m. Finally, it is worth noting that the numerical implementation of this method for three-dimensional problems can be achieved with relative ease by combining (existing) finite element programs and customized optimizing codes.

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